Lecture # 13

Last time: X a discrete random var. with probability mass function $p(x)$.

$$\mu = E[X] = \sum_{x:\, p(x) > 0} x\, p(x)$$ the mean/average/expected value.

and $$Var(X) = E[(X-\mu)^2]$$
$$= E[X^2] - (E[X])^2$$

Next we want to consider some specific DRV's

Bernoulli / Binomial:

Defn: We say that a DRV is Bernoulli if range $(X) = \{0, 1\}$ (think: success and failure). and X has a probability mass function

$$p(0) = P(X=0) = 1-p \quad \text{(failure).}$$
$$p(1) = P(X=1) = p \quad \text{(success.)}$$

(called a Bernoulli trial).

Ex: Tossing a coin $\{h, t\}$.

$$X : \{h, t\} \to \{0, 1\}.$$

$$X(h) = 0 \qquad p = 1 - p = 1/2.$$
$$X(t) = 1$$

Rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$.

$$X : S \to \{0, 1\}.$$

$$X(x) = \begin{cases} 0 & \text{iff } x = 1 \\ 1 & \text{ow}. \end{cases}$$

$$P(X = 0) = 1/6 \qquad P(X = 1) = 5/6.$$

Suppose we fix $n \geq 1$ and set up an experiment that consists of n independent Bernoulli trials, each having success with some fixed probability p. Let X be the DRV given by

$$X = \# \text{ of successes in } n \text{ trials}.$$

Then we say that X is a Binomial RV with parameters n, p. (Sometimes write $X = Bin(n, p)$)
$\text{range}(X) = \{0, \dots, n\}$.

What is the pmf of X? Suppose we want to know $P(X=i)$. Given a particular sequence of i successes and $n-i$ failures, (e.g. maybe i successes first followed by $n-i$ failures), then the probability of that particular sequence is

$$p^i (1-p)^{n-i}$$

i successes $n-i$ failures.

Now, since i successes can appear in any of the $\binom{n}{i}$ many ways, and each of those ways is equally likely, we have

$$p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}.$$

<u>Ex:</u> Suppose you have a test coming up which consists of 25 multiple choice questions, each which are independent, with four possible answers and only one right answer. Suppose you didn't study and are going to guess randomly.

What is your probability of passing??

Note that each question is a Bernoulli trial with $p = 1/4$.

Suppose that $X = \#$ of correct questions.

If 50% is a pass, then we want to compute

$$P(X \geq 12.5) = \underbrace{P(X \geq 13)}_{\text{since } X \text{ is discrete.}}$$

We use the cumulative distribution function

$$P(X \geq 12.5) = P(X \geq 13) = \sum_{i=13}^{25} \binom{25}{i} \left(\frac{1}{4}\right)^i \left(\frac{3}{4}\right)^{25-i}$$

$$\approx 0.0034 = 0.34\% \text{ chance of passing } \ddot{-}$$

<u>Properties of the Binomial Random var.</u>

<u>Expected value and Variance:</u> Let $X = Bin(n, p)$.

Let $k \geq 1$. We have

$$E[X^k] = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}.$$

Observe that

$$\underbrace{n-i+i-1}_{=n-1}$$

$$i\binom{n}{i} = i\,\frac{n!}{i!\,(n-i)!} = \frac{n\,(n-1)!}{(i-1)!\,(n-i)!}$$
$$= n\binom{n-1}{i-1}.$$

So
$$E[X^k] = \sum_{i=1}^{n} n\,i^{k-1}\binom{n-1}{i-1}p^i(1-p)^{n-i}$$
$$= np\sum_{i=1}^{n} i^{k-1}\binom{n-1}{i-1}p^{i-1}(1-p)^{n-i}$$

Setting $j = i-1$, we have
$$E[X^k] = np\sum_{j=0}^{n-1}(j+1)^{k-1}\binom{n-1}{j}p^j(1-p)^{n-1-j}$$
$$= np\,E\left[(\mathrm{Bin}(n-1,p)+1)^{k-1}\right]$$

If we set $k=1$, then
$$E[X] = np\,E\left[(\mathrm{Bin}(n-1,p)+1)^{0}\right] = \underline{np}.$$

If we set $k=2$, then
$$E[X^2] = np\,E[\mathrm{Bin}(n-1,p)+1]$$
$$= np\left(E[\mathrm{Bin}(n-1,p)]+1\right)$$
$$= np\,((n-1)p+1).$$

$$= np^2(n-1) + np.$$

So $Var(X) = E[X^2] - (E[X])^2$

$$= np((n-1)p + 1) - (np)^2$$

$$= np[\cancel{np} - p + 1 - \cancel{np}]$$

$$Var(X) = np(1-p).$$

<u>Ex</u>: In the previous example, we had $X = Bin(25, \frac{1}{4})$ so, if everyone were to guess on their tests, we would expect the class average to be

$$E[X] = \frac{25}{4} = 6.25/25. \text{ i.e. } 25\%$$

which makes sense.

$$Var(X) = 4.69.$$

The following recursion is useful for computing binomial distributions. $X = Bin(n,p)$ then

$$P(X=k+1) = \frac{p}{1-p}\frac{n-k}{k+1}P(X=k).$$